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Numerical Simulation of Continuous-Time Stochastic Dynamical Systems with Noisy Measurements and Their Discrete-Time Equivalents

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Abstract— **Computer-aided control system design software packages offer several numerical methods to simulate continuous-time (CT) deterministic dynamical systems. However, simulating stochastic dynamical systems (SDSs) has not received as much attention. This paper presents a self-contained treatment of how to simulate linear time-invariant CT SDSs with noisy measurements. The system model considered is quite general and avoids many simplifying assumptions generally made in the literature. An analytical solution to a state-space representation of the CT SDS is presented, and numerical approximations to such solution are derived. Moreover, the connection between the CT SDS and a discrete-time (DT) statistically equivalent system is established. Numerical methods for computing the various DT system and noise statistics matrices are discussed. An example is presented to illustrate the application of the various methods discussed in this paper.**

I. INTRODUCTION

Computer-aided control system design (CACSD) software packages have paid considerable attention to creating simulation environments for dynamical systems, whose dynamics evolve in a deterministic fashion according to ordinary differential equations (ODEs). Such simulation environments often accommodate systems with continuous-time (CT) dynamics, discrete-time (DT) dynamics, or a hybrid of both. The system dynamics can be composed through basic building blocks (e.g. integrators, discrete delays, summation nodes) or through built-in routines for common system representations (e.g. transfer function and state-space). Moreover, such environments offer several fixed and variable step-size numerical solvers to approximate solutions of the ODEs (e.g. Euler, Runge-Kutta, Adams-Moulton, and Rosenbrock) [1], [2].

Deterministic models have several shortcomings. On one hand, the system dynamics are usually not only driven by prescribed inputs, but also by disturbances, which can neither be prescribed nor modeled deterministically. Also, actuator noise may arise in the translation of the known computed inputs into the physical signals applied to the system. Moreover, there may exist uncertainty in the system dynamics due to neglected effects or modeling mismatches. Such uncertainties are usually modeled by a stochastic process acting on the system dynamics, often referred to as process noise. On the other hand, sensors usually do not yield exact readings of the desired quantities, but introduce their own dynamics and distortions. Such uncertainties are usually modeled by a stochastic process acting on the measurements, often referred to as measurement noise. Stochastic dynamical systems (SDSs) whose dynamics are governed by stochastic differential equations (SDEs) and whose measurements are noisy have been playing an increasingly prominent role in modeling a range of application areas of interest to control systems designers, such as microelectronics, mechanics, navigation, biology, chemistry, economics, and finance.

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In this paper, a SDS model is adopted, which is quite general and avoids many simplifying assumptions generally made in the literature. In this respect, the state dynamics will be modeled as a linear-time invariant (LTI) SDE that is driven by additive known inputs and process noise. The measurements will be modeled as LTI with direct additive feedthrough of the known inputs. The measurements are corrupted by two additive, correlated, non-zero mean noise components, the process noise and the measurement noise. It will be assumed that the process and measurement noise are Gaussian and temporally uncorrelated. Extensions to the case where the noise components are non-Gaussian and/or temporally correlated can be readily achieved by introducing proper filters and state augmentation [3]. The inclusion of the process noise in the measurement model adds a degree of freedom in the modeling procedure, and is used to model situations where the process noise directly affects the measurements or in the case of noisy actuator with direct feedthrough from the inputs to the measurements.

Since common CACSD packages do not contain built-in routines to simulate SDSs, control systems designers tend to adapt existing ODE solvers to simulate SDEs. In this respect, a common pitfall is committed, in which the process noise is generated, added to the state and the known input, then fed to the integrator. If the underlying ODE solver has variable-step size and the process noise generation routine executes at minor time-steps of the solver, then the solver may get into the following mode. As noise samples are generated at minor time-steps, and if the noise intensity is significant, the solver will start taking increasingly smaller steps to meet the convergence criteria, which may result in very slow simulations and even erroneous termination of the solver. To circumvent this problem, a common approach is to specify the execution of the process noise generation so it executes at major time-steps only. Even in this case, if the CT process noise statistics are not transformed correctly before calling the process noise generator, then the generated numerical solution will not correspond to the desired SDE.

A complete understanding of numerical simulations of SDEs requires familiarity with advanced probability and stochastic processes, which might be formidable to many control systems designers [4], [5]. This paper discusses how simulate SDSs in a language comprehensible to control systems designers. The discussions in this paper are self-contained in the sense that the developed algorithms will make use of basic building blocks that are readily available in CACSD packages, such as matrix exponentials, scalar random number generators, and discrete unit delays.

The problem of discretizing CT LTI SDSs with noisy measurements is also considered in this paper. The treatment of this subject has been addressed in the literature [3], [6], but the models considered suffer from the simplifying assumptions, which are relaxed in this paper. One application that necessitates achieving such discretization is digital implementation of control and estimation algorithms, such as linear quadratic Gaussians (LQG) and Kalman filters (KFs). In particular, DT recursive KFs require solving a difference Riccati equation, whereas the CT Kalman-Bucy filter requires solving a differential Riccati equation. Difference Riccati equation solvers are significantly easier and cheaper to implement than differential Riccati equations [7].

This paper is organized as follows. Section II describes the SDS model considered in this paper. Section III presents the analytical solution of the SDE, derives the Euler-Maruyama method for simulating the SDE numerically, and discusses the method's convergence. Section IV derives a DT equivalent to the CT SDS and establishes connections between their noise statistics. Section V presents numerical algorithms to compute the DT system and noise statistics matrices. Section VI outlines how to generate samples of non-zero mean, spatially-correlated process and measurement noise vectors. Section VII applies the various methods presented in this paper into simulating a CT SDS and its DT equivalent. Concluding remarks are discussed in Section VIII.

II. Stochastic Dynamical System Model

This section describes the SDS model considered in this paper. The CT SDS is governed by the Langevin equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{w}(t), \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times r}$, $\mathbf{G} \in \mathbb{R}^{n\times q}$, $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state vector, $\mathbf{u}(t) \in \mathbb{R}^r$ is a known input signal, and $\mathbf{w}(t) \in \mathbb{R}^q$ is the process noise vector, which is modeled as a CT temporally uncorrelated Gaussian stochastic process with first- and second-order statistics given by

$$\mathbb{E}\left[\mathbf{w}(t)\right] = \mathbf{m}_{\mathbf{w}}(t),\ \operatorname{cov}\left[\mathbf{w}(t), \mathbf{w}(\tau)\right] = \mathbf{Q}(t)\delta\left(t-\tau\right), \tag{2}$$

where $\mathbb{E}[\cdot]$ is the expectation, cov$[\cdot]$ is the covariance, and $\delta(\cdot)$ is the Dirac delta function.

The CT SDS (12) will be interpreted by the $n-$dimensional vector-valued Itô SDE

$$d\mathbf{x}(t) = \left[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{m}_{\mathbf{w}}(t)\right]dt + \tilde{\mathbf{G}}(t)d\mathbf{W}(t), \tag{3}$$

where $t \in [t_0, t_f]$, $\tilde{\mathbf{G}}(t) = \mathbf{G}\mathbf{U}(t)\mathbf{\Lambda}^{\frac{1}{2}}(t)$, $\mathbf{U}(t)$ and $\mathbf{\Lambda}(t)$ correspond to the eigen-decomposition of $\mathbf{Q}(t)$ such that $\mathbf{Q}(t) = \mathbf{U}(t)\Lambda(t)\mathbf{U}^{\mathsf{T}}(t)$. The incremental Wiener process $d\mathbf{W}(t)$ is defined as $d\mathbf{W}(t) = \tilde{\mathbf{w}}(t)dt$, where $\tilde{\mathbf{w}}(t) \triangleq \mathbf{w}(t) - \mathbf{m}_{\mathbf{w}}(t)$, with $\mathbf{W}(t) = \left[W_1(t), \ldots, W_q(t)\right]^{\mathsf{T}}$, and $\{W_i(t)\}_{i=1}^{q}$ are independent standard scalar Wiener processes with respect to a common family of $\sigma-$algebras $\{\mathcal{A}(t), t \geq 0\}$. A standard Wiener process (Brownian motion) $\{W(t), t \in [0, t_f]\}$ is a Gaussian process with independent increments, such that

- $W(0) = 0$ with probability 1, and $\mathbb{E}\left[W(t)\right] = 0$.
- For $0 \leq s < t \leq t_f$, the random variable given by the increment $[W(t) - W(s)]$ is Gaussian with $[W(t) - W(s)] \sim \sqrt{t-s}\mathcal{N}(0,1)$.
- For $0 \leq s < t < u < v \leq t_f$, the increments $[W(t) - W(s)]$ and $[W(v) - W(u)]$ are independent.

Note that while (1) is common in representing SDSs, it is actually an abuse of notation, since sample paths of $W(t)$ are continuous everywhere, differentiable nowhere with probability 1, and have unbounded variations on any finite subinterval [8].

The measurements made on the SDS (1) are according to

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{H}\mathbf{w}(t) + \mathbf{v}(t), \tag{4}$$

where $\mathbf{C} \in \mathbb{R}^{m\times n}$, $\mathbf{D} \in \mathbb{R}^{m\times r}$, and $\mathbf{H} \in \mathbb{R}^{m\times q}$, $\mathbf{y}(t) \in \mathbb{R}^m$ is the observation vector, and $\mathbf{v}(t) \in \mathbb{R}^m$ is the measurement noise vector, which is modeled as a CT temporally uncorrelated Gaussian stochastic process with first- and second-order statistics given by

$$\mathbf{E}\left[\mathbf{v}(t)\right] = \mathbf{m}_{\mathbf{v}}(t),\ \operatorname{cov}\left[\mathbf{v}(t), \mathbf{v}(\tau)\right] = \mathbf{R}(t)\delta\left(t-\tau\right). \tag{5}$$

The cross-covariance between the process and measurement noise will be such that

$$\operatorname{cov}\left[\mathbf{w}(t), \mathbf{v}(\tau)\right] = \mathbf{N}(t)\delta\left(t-\tau\right). \tag{6}$$

The observations (4) will be similarly interpreted as

$$\begin{aligned} d\mathbf{z}(t) \triangleq \mathbf{y}(t)dt = &\left[\mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{H}\mathbf{m}_{\mathbf{w}}(t) + \mathbf{m}_{\mathbf{v}}(t)\right]dt \\ &+ \mathbf{H}d\mathbf{W}(t) + d\mathbf{V}(t), \end{aligned} \tag{7}$$

where $d\mathbf{V}(t) = \tilde{\mathbf{v}}(t)dt$ and $\tilde{\mathbf{v}}(t) \triangleq \mathbf{v}(t) - \mathbf{m}_{\mathbf{v}}(t)$.

III. Numerical Simulation of Continuous-Time Stochastic Dynamical Systems

A. Analytical Solution to Linear SDEs

While the SDE in (3) is written as a stochastic differential, it is interpreted as a stochastic integral, because the solution inherits the non-differentiability of the sample paths of $\mathbf{W}(t)$. In order for the solution $\mathbf{x}(t)$ to be non-anticipative with respect to $\mathbf{W}(t)$, the restriction that the initial condition $\mathbf{x}(t_0)$ be independent of $\{\mathbf{W}(t), t \in [t_0, t_f]\}$, is imposed.

As long as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{G}$, and $\tilde{\mathbf{G}}(t)$ are Lebesgue-measurable and bounded on $t \in [t_0, t_f]$, the existence and uniqueness theorems apply for (3) ensuring the existence of a strong solution $\mathbf{x}(t)$ on $[t_0, t_f]$ for each $\mathcal{A}(t_0)$-measurable initial condition $\mathbf{x}(t_0)$ corresponding to a given Wiener process $\{\mathbf{W}(t), t \in [t_0, t_f]\}$ and associated family of $\sigma-$algebras. Moreover, the solution is a homogeneous Markov process.

The solution to (3) can be expressed analytically as

$$\mathbf{x}(t) = \mathbf{\Phi}_{t,t_0}\left[\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{\Phi}_{\tau,t_0}^{-1}\left[\mathbf{B}\mathbf{u}(\tau) + \mathbf{G}\mathbf{m}_{\mathbf{w}}(\tau)\right]d\tau\right] + \mathbf{\Phi}_{t,t_0}\left[\sum_{i=1}^{q}\int_{t_0}^{t}\mathbf{\Phi}_{\tau,t_0}^{-1}\mathbf{G}_i(\tau)dW_i(\tau)\right], \quad (8)$$

where, $\mathbf{\Phi}_{t,t_0} = e^{\mathbf{A}(t-t_0)}$ is the fundamental solution and $\tilde{\mathbf{G}}_i(t)$ is the i^{th} column vector of $\tilde{\mathbf{G}}(t)$. The first integral is pathwise a deterministic Riemann integral and the second is an Itô stochastic integral. Whenever the initial condition $\mathbf{x}(t_0)$ is a constant or a Gaussian random vector, the solution (8) is a Gaussian process.

B. Numerical Approximations to Solutions of Linear SDEs

Considerable care must be taken while deriving numerical schemes for approximating the solution to (8). This is due to the fact that while the integrand function in a Riemann sum approximating the Riemann integral can be evaluated at an arbitrary point of the discretization subinterval, for the Itô stochastic integral, the integrand function must always be evaluated at the left-hand endpoints. For instance, adapting the deterministic Heun scheme (second order Runge-Kutta scheme) to the Itô SDE results in a solution that is not consistent with Itô calculus, and so does not converge in the weak nor strong senses [8].

In this paper, the Euler-Maruyama (EM) scheme will be derived for (3). For a given discretization of the time interval $[t_0, t_f]$ such that $t_0 \equiv 0 < T < 2T < \ldots < NT \equiv t_f$, the EM method yields a CT stochastic process $\hat{\mathbf{x}} = \{\hat{\mathbf{x}}(t), t_0 \le t \le t_f\}$ satisfying

$$\hat{\mathbf{x}}\left[(k+1)T\right] = \hat{\mathbf{x}}(kT) + \left[\mathbf{A}\hat{\mathbf{x}}(kT) + \mathbf{B}\mathbf{u}(kT) + \mathbf{G}\mathbf{m}_{\mathbf{w}}(kT)\right]T + \sum_{i=1}^{q}\tilde{\mathbf{G}}_i(kT)\Delta W_i(kT), \quad k \in \mathbb{Z} \quad (9)$$

where $\hat{\mathbf{z}}_0 = \mathbf{x}_0$, T is the sampling period, and $\Delta W_i(kT) = [W_i[(k+1)T] - W_i(kT)] \sim \mathcal{N}(0,T)$. The random increments $\Delta W_i(kT)$ and $\Delta W_j(kT)$ are independent $\forall i \neq j$ and can be generated through a sequence of independent Gaussian random number generators.

The Milstein scheme is very similar to the EM scheme except for the addition of an extra term, which is a function of the diffusion term $\tilde{\mathbf{G}}(t)$, its partial derivatives with respect to the state vector $\mathbf{x}$, random increments, and the sampling period T. Since the diffusion term in the SDE model (3) is not a function of $\mathbf{x}$, it can be shown that the Milstein additional term is zero, in which case the derived EM scheme (9) coincides with the Milstein scheme.

C. Convergence of Numerical Approximations

Generally, the accuracy of DT approximations of the solution (8) can be judged in terms of the convergence of the numerical solution, leading to the notions of strong and weak convergence. Methods with strong convergence produce pathwise approximations. They are used in direct simulations, filtering, and testing statistical estimators. Methods with weak convergence produce approximating expectations of functionals of the Itô process, such as its probability distribution and its moments. They are used in problems when such functionals cannot be determined analytically. A DT approximation $\hat{\mathbf{x}}$ is said to converge strongly to $\mathbf{x}$ with order $\gamma > 0$ at time T, if there exists $C > 0$, which does not depend on the maximum step size $\delta \in (0, \delta_0)$ for some $\delta_0 > 0$ such that

$$\varepsilon(\delta) = \mathbb{E}\left[\|\mathbf{x}(T) - \hat{\mathbf{x}}(T)\|\right] \le C\delta^{\gamma}.$$

The EM method is known to have a strong order of convergence $\gamma = \frac{1}{2}$ and gives good numerical results whenever the drift and diffusion coefficients are nearly constant. Since the EM method derived coincides with the Milstein scheme, its order of convergence becomes $\gamma = 1$. Other numerical methods for approximating solutions to SDEs with varying orders of convergence include the Order 1.5 and Order 2.0 Taylor schemes and Runge-Kutta scheme [4].

IV. Discretization of Continuous-Time Stochastic Dynamical Systems

This section derives DT equivalent to the CT SDS (1), (4) and establishes connections between their noise statistics.

Theorem 1: Given the SDS (1) with measurements made according to (4). Assume that the input signal $\mathbf{u}$, the process noise mean vector $\mathbf{m}_{\mathbf{w}}$, and the measurement noise mean vector $\mathbf{m}_{\mathbf{v}}$ take piecewise constant values within the sampling period T, i.e.

$$\mathbf{u}(t) = \mathbf{u}(kT),\ \mathbf{m}_{\mathbf{w}}(t) = \mathbf{m}_{\mathbf{w}}(kT),\ \mathbf{m}_{\mathbf{v}}(t) = \mathbf{m}_{\mathbf{v}}(kT),$$

$kT \le t < (k+1)T$, $k \in \mathbb{Z}$. Moreover, assume that the process and measurement noise auto-covariances and cross-covariance to be stationary within T, i.e.

$$\mathbf{Q}(t) = \mathbf{Q}(kT),\ \mathbf{R}(t) = \mathbf{R}(kT),\ \mathbf{N}(t) = \mathbf{N}(kT),$$

$t \in [kT, (k+1)T)$. Then, the DT equivalent of the state dynamics is given by

$$\mathbf{x}\left[(k+1)T\right] = \mathbf{A}_d\mathbf{x}(kT) + \mathbf{B}_d\mathbf{u}(kT) + \mathbf{n}(kT), \quad (10)$$

$$\mathbf{A}_d \triangleq e^{\mathbf{A}T},\ \mathbf{B}_d \triangleq \int_0^T e^{\mathbf{A}\eta}d\eta\mathbf{B}. \quad (11)$$

The first- and second-order statics of the DT noise sequence $\mathbf{n}(kT)$ are given by

$$\mathbb{E}\left[\mathbf{n}(kT)\right] = \mathbf{G}_d\mathbf{m}_{\mathbf{w}}(kT)$$
$$\text{cov}\left[\mathbf{n}(kT), \mathbf{n}(lT)\right] = \mathbf{Q}_d(kT)\delta\left[kT - lT\right],$$

where $\delta\left[\cdot\right]$ is the Kronecker delta function and

$$\mathbf{G}_d \triangleq \int_0^T e^{\mathbf{A}\eta}d\eta\,\mathbf{G},\ \ \mathbf{Q}_d(kT) \triangleq \int_0^T e^{\mathbf{A}\eta}\mathbf{G}\mathbf{Q}(kT)\mathbf{G}^{\mathsf{T}}e^{\mathbf{A}^{\mathsf{T}}\eta}d\eta.$$

The DT equivalent of the measurement equation is given by

$$\mathbf{y}(kT) = \mathbf{C}_d\mathbf{x}(kT) + \mathbf{D}_d\mathbf{u}(kT) + \mathbf{e}(kT),$$

where $\mathbf{C}_d \triangleq \mathbf{C}$, $\mathbf{D}_d \triangleq \mathbf{D}$, and the first- and second-order statics of the DT noise sequence $\mathbf{e}(kT)$ are given by

$$\begin{aligned}\mathbb{E}\left[\mathbf{e}(kT)\right] &= \mathbf{H}\mathbf{m}_{\mathbf{w}}(kT) + \mathbf{m}_{\mathbf{v}}(kT)\\ \mathrm{cov}\left[\mathbf{e}(kT), \mathbf{e}(lT)\right] &= \mathbf{R}_d(kT)\delta\left[kT - lT\right]\\ \mathrm{cov}\left[\mathbf{n}(kT), \mathbf{e}(lT)\right] &= \mathbf{N}_d(kT)\delta\left[kT - lT\right]\end{aligned}$$

$$\mathbf{R}_d(kT) \triangleq \frac{1}{T}\left[\mathbf{H}\mathbf{Q}(kT)\mathbf{H}^{\mathsf{T}} + \mathbf{H}\mathbf{N}(kT) + \mathbf{N}^{\mathsf{T}}(kT)\mathbf{H}^{\mathsf{T}} + \mathbf{R}(kT)\right]$$

$$\mathbf{N}_d(kT) \triangleq \frac{1}{T}\left[\int_0^T e^{\mathbf{A}\eta}d\eta\left[\mathbf{G}\mathbf{Q}(kT)\mathbf{H}^{\mathsf{T}} + \mathbf{G}\mathbf{N}(kT)\right]\right].$$

Proof: First, consider the DT equivalent of the state dynamics. The SDS (1) can be readily transformed such that the noise is zero-mean as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{G}\mathbf{m}_{\mathbf{w}}(t) + \mathbf{G}\tilde{\mathbf{w}}(t), \qquad (12)$$

where $\tilde{\mathbf{w}}(t) \triangleq \mathbf{w}(t) - \mathbf{m}_{\mathbf{w}}(t)$ is a zero-mean white Gaussian stochastic process with auto-covariance $\mathbf{Q}(t)\delta\left(t-\tau\right)$. The solution to the SDS (12) is given by

$$\begin{aligned}\mathbf{x}(t) \;=\; & e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\left[\mathbf{B}\mathbf{u}(\tau) + \mathbf{G}\mathbf{m}_{\mathbf{w}}(\tau)\right]d\tau\\ & + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{G}\tilde{\mathbf{w}}(\tau)d\tau. \qquad (13)\end{aligned}$$

Letting $t_0 \triangleq kT$ and $t \triangleq (k+1)T$ in (13), using the fact that $\mathbf{u}$ and $\mathbf{m}_{\mathbf{w}}$ take piecewise constant values within T, and using the change of variable $\eta = (k+1)T - \tau$ yields

$$\mathbf{x}\left[(k+1)T\right] = \mathbf{A}_d\mathbf{x}(kT) + \mathbf{B}_d\mathbf{u}(kT) + \mathbf{n}(kT), \qquad (14)$$

$$\mathbf{n}(kT) \triangleq \qquad (15)$$
$$\int_0^T e^{\mathbf{A}\eta}d\eta\mathbf{G}\mathbf{m}_{\mathbf{w}}(kT) + \int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}\tilde{\mathbf{G}}(\tau)\tilde{\mathbf{w}}(\tau)d\tau.$$

The first-order statistics can be shown by direct substitution and noting that $\tilde{\mathbf{w}}$ has zero-mean and that $\mathbf{m}_{\mathbf{w}}$ is constant within T. The second-order statistics can be shown to be

$$\mathrm{cov}\left[\mathbf{n}(kT), \mathbf{n}(lT)\right] = \qquad (16)$$
$$\int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}\mathbf{G}\mathbf{Q}(\tau)\mathbf{G}^{\mathsf{T}}e^{\mathbf{A}^{\mathsf{T}}[(k+1)T-\tau]}d\tau\delta\left[kT - lT\right].$$

Equation (16) was derived by direct substitution and noting that $\tilde{\mathbf{w}}$ is zero-mean. Then, by employing the properties of the Dirac delta function, namely that it is even and the sifting property, and noting that on the support of $\tau \in [kT, (k+1)T)$, the only time the integrand has a non-zero value is whenever $kT \equiv lT$.

Finally, since $\mathbf{Q}$ is stationary within T, and using the change of variables $\eta = (k+1)T - \tau$ in (16), the second-order statistics for $\mathbf{n}$ follows.

The DT equivalent to (4) is considered next. The measurement equation (4) can be transformed such that the process and measurement noises are zero-mean as

$$\begin{aligned}\mathbf{y}(t) \;&=\; \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{e}(t) \qquad (17)\\ \mathbf{e}(t) \;&\triangleq\; \mathbf{H}\mathbf{m}_{\mathbf{w}}(t) + \mathbf{m}_{\mathbf{v}}(t) + \mathbf{H}\tilde{\mathbf{w}}(t) + \tilde{\mathbf{v}}(t), \qquad (18)\end{aligned}$$

where $\tilde{\mathbf{w}}(t) \triangleq \mathbf{v}(t) - \mathbf{m}_{\mathbf{v}}(t)$ is a zero-mean white Gaussian stochastic process with auto-covariance $\mathbf{R}(t)\delta\left(t-\tau\right)$. The cross-covariance between $\tilde{\mathbf{w}}(t)$ and $\tilde{\mathbf{v}}(t)$ is $\mathbf{N}(t)\delta\left(t-\tau\right)$. It makes no sense to sample the measurement $\mathbf{y}(t) = d\mathbf{z}(t)/dt$ directly, as this will lead to a DT system with infinite variance. This problem is typically resolved by replacing the impractical ideal sampler by a practical form of sampling in which the signal is passed through a filter prior to sampling [7]. To this end, the DT equivalent of the measurement signal will be viewed as a short-time average [6] according to

$$\mathbf{y}(kT) = \mathbf{C}_d\mathbf{x}(kT) + \mathbf{D}_d\mathbf{u}(kT) + \mathbf{e}(kT), \qquad (19)$$

where $\mathbf{C}_d = \mathbf{C}$, $\mathbf{D}_d = \mathbf{D}$, and $\mathbf{e}(kT)$ is defined by

$$\mathbf{e}(kT) \triangleq \frac{1}{T}\int_{kT}^{(k+1)T}\mathbf{e}(\tau)d\tau.$$

The first-order statistics of $\mathbf{e}(kT)$ can be shown by direct substitution and noting that $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{v}}$ have zero mean and that $\mathbf{m}_{\mathbf{w}}$ and $\mathbf{m}_{\mathbf{v}}$ take piecewise constant values within T. Next, after some manipulations, the second-order statistics of $\mathbf{e}(kT)$ can be shown to be

$$\begin{aligned}\mathrm{cov}\left[\mathbf{e}(kT), \mathbf{e}(lT)\right] = \frac{1}{T^2}\int_{kT}^{(k+1)T}\int_{lT}^{(l+1)T} & \left[\mathbf{H}\mathbf{Q}(\tau)\mathbf{H}^{\mathsf{T}} + \mathbf{H}\mathbf{N}(\tau)\right.\\ & \left. + \mathbf{N}^{\mathsf{T}}(\tau)\mathbf{H}^{\mathsf{T}} + \mathbf{R}(\tau)\right]\delta(\tau - \eta)d\eta d\tau. \qquad (20)\end{aligned}$$

By using the evenness and sifting properties of the dirac delta function, and noting that on the support of $\tau \in [kT, (k+1)T)$, the only time the integrand has a non-zero value is whenever $kT \equiv lT$, the inner integral can be solved to get

$$\begin{aligned}\mathrm{cov}\left[\mathbf{e}(kT), \mathbf{e}(lT)\right] = \frac{1}{T^2}\int_{kT}^{(k+1)T} & \left[\mathbf{H}\mathbf{Q}(\tau)\mathbf{H}^{\mathsf{T}} + \mathbf{H}\mathbf{N}(\tau)\right.\\ & \left. + \mathbf{N}^{\mathsf{T}}(\tau)\mathbf{H}^{\mathsf{T}} + \mathbf{R}(\tau)\right]\delta\left[kT - lt\right]d\tau.\end{aligned}$$

Using the fact that $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{N}$ are constant within T allows us to bring the integrand outside the integral and yields the final expression for $\mathrm{cov}\left[\mathbf{e}(kT), \mathbf{e}(lT)\right]$.

Finally, the cross-covariance between $\mathbf{n}$ and $\mathbf{e}$ can established by substituting for $\mathbf{n}$ and $\mathbf{e}$, employing similar arguments used in deriving the DT auto-covariances and mean vectors of $\mathbf{n}$ and $\mathbf{e}$, and after some manipulations, the final expression for $\mathrm{cov}\left[\mathbf{n}(kT), \mathbf{e}(lT)\right]$ results. ■

V. Computing Discrete-Time Equivalent System Matrices and Noise Statistics

This section presents numerical methods for computing the DT equivalent system matrices and noise statistics. By analyzing the various DT system matrices and first- and second-order statistics derived in Section IV, we note that they involve terms of the form

$$e^{\mathbf{A}T}, \quad \int_0^T e^{\mathbf{A}\eta}d\eta\,\mathbf{M}, \quad \int_0^T e^{\mathbf{A}\eta}\,\mathbf{N}\,e^{\mathbf{A}^{\mathsf{T}}\eta}d\eta, \qquad (21)$$

for some $\mathbf{M} \in \mathbb{R}^{n\times p}$ and $\mathbf{N} \in \mathbb{R}^{n\times n}$. The first term is the "classic" matrix exponential, to which several numerical algorithms have been developed [9]. CACSD packages possess built-in routines that can readily compute this term.

The second term's solution has the form $\left[e^{\mathbf{A}T} - \mathbf{I}\right] \mathbf{A}^{-1}\mathbf{M}$. A numerical solution to evaluate the second term that avoids the matrix inversion, which would be problematic if $\mathbf{A}$ is singular, can be constructed as follows. For some matrix $\mathbf{Z} \in \mathbb{R}^{n\times n}$ and a scalar T, the Taylor Series Expansion (TSE) of the matrix exponential $e^{\mathbf{Z}T}$ is given by

$$e^{\mathbf{Z}T} = \mathbf{I} + \mathbf{Z}T + \mathbf{Z}^2\frac{T^2}{2!} + \mathbf{A}^3\frac{T^3}{3!} + \ldots = \sum_{i=0}^{\infty} \frac{(\mathbf{Z}T)^i}{i!}. \quad (22)$$

Next, constructing the matrix $\mathbf{Z} \triangleq \begin{bmatrix} \mathbf{A} & \mathbf{M} \\ \mathbf{0}_{p\times n} & \mathbf{0}_{p\times p} \end{bmatrix}$, we note that $\mathbf{Z}^i \triangleq \begin{bmatrix} \mathbf{A}^i & \mathbf{A}^{i-1}\mathbf{M} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. Hence, the matrix exponential can be expressed as

$$e^{\mathbf{Z}T} = \begin{bmatrix} e^{\mathbf{A}T} & \left(\sum_{i=1}^{\infty} \frac{\mathbf{A}^i}{i!}\right) \mathbf{A}^{-1}\mathbf{M} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Finally, we note that the 2×1 block, $\left(\sum_{i=1}^{\infty} \frac{\mathbf{A}^i}{i!}\right) \mathbf{A}^{-1}\mathbf{M} = \left[e^{\mathbf{A}T} - \mathbf{I}\right] \mathbf{A}^{-1}\mathbf{M}$. Hence, compact evaluation of the first and second terms in (21) can be achieved by computing the matrix exponential $e^{\mathbf{Z}T}$.

Two methods to evaluate the third term in (21) are presented. If the sampling period is relatively small, the TSE in (22) can be truncated to include the first term only, i.e. the identity matrix. Consequently, the third term in (21) can be approximated as $\mathbf{N}T$.

The second method is based on Van Loan's algorithm [10], [11], in which a numerical algorithm for solving four classes of integrals involving matrix exponentials was proposed. In this respect, the evaluation of the third term can be achieved by defining the matrix

$$\exp\left\{\begin{bmatrix} -\mathbf{A} & \mathbf{0}_{n\times r} & \mathbf{N} & \mathbf{0}_{n\times r} \\ \mathbf{0}_{r\times n} & \mathbf{0}_{r\times r} & \mathbf{0}_{r\times n} & \mathbf{0}_{r\times r} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times r} & \mathbf{A}^{\mathsf{T}} & \mathbf{0}_{n\times r} \\ \mathbf{0}_{r\times n} & \mathbf{0}_{r\times r} & \mathbf{0}_{r\times n} & \mathbf{0}_{r\times r} \end{bmatrix} T\right\} \triangleq \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{\Phi}_{12} \\ \mathbf{0} & \mathbf{\Phi}_{22} \end{bmatrix}.$$

It turns out that

$$\int_0^T \begin{bmatrix} e^{\mathbf{A}\eta}\mathbf{N}e^{\mathbf{A}^{\mathsf{T}}\eta} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} d\eta = \Phi_{22}^{\mathsf{T}}\Phi_{12}. \quad (23)$$

VI. Generating Samples of Process and Measurement Noise Vectors

In simulating SDS, one must generate samples of the random vectors corresponding to the process and measurement noise, according to specified statistics. Most random number generators in CACSD packages can generate samples of Gaussian-distributed random variables with a specified mean and variance. To generate samples of random vectors whose elements are uncorrelated, simple concatenation of the outputs of such random number generators suffice. However, generating samples of random vectors whose elements are correlated using these generators is not straightforward. This section presents how to generate samples of Gaussian-distributed random vectors with specified statistics.

Proposition 1: Given a random number generator that generates samples of the random variable $Y \sim \mathcal{N}(0, 1)$. Define the $n-$dimensional random vector $\mathbf{Y}$ whose elements are iid with $Y_i \sim \mathcal{N}(0, 1)$; hence, $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. Then, generating samples of the random vector $\mathbf{X} \sim \mathcal{N}(\mathbf{m_X}, \mathbf{C_{XX}})$ can be accomplished by the affine transformation $\mathbf{X} = \mathbf{AY} + \mathbf{b}$, where $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}^{\frac{1}{2}}$, $\mathbf{b} = \mathbf{m_X}$, $\mathbf{V}$ is a matrix whose columns are the eigenvectors of $\mathbf{C_{XX}}$, and $\mathbf{\Lambda}$ is a diagonal matrix whose elements are the eigenvalues of $\mathbf{C_{XX}}$.

Proof: The proof can be established by direct substitution and noting that through the eigenvalue decomposition, we can express $\mathbf{C_{XX}} = \mathbf{V\Lambda V}^{\mathsf{T}}$.

■

Corollary 1: Generating samples of the $n-$dimensional vector $\mathbf{w}$ and the $m-$dimensional vector $\mathbf{v}$, such that $\mathbf{w} \sim \mathcal{N}(\mathbf{m_w}, \mathbf{Q})$, $\mathbf{v} \sim \mathcal{N}(\mathbf{m_v}, \mathbf{R})$, and the cross cross-covariance between $\mathbf{v}$ and $\mathbf{v}$ is $\mathbf{N}$, can be accomplished by generating samples of the random vector $\mathbf{Z} = \left[\mathbf{w}^{\mathsf{T}}\, \mathbf{v}^{\mathsf{T}}\right]^{\mathsf{T}}$ with $\mathbf{Z} \sim \mathcal{N}(\mathbf{m_Z}, \mathbf{C_{ZZ}})$, where

$$\mathbf{m_Z} = \begin{bmatrix} \mathbf{m_w} \\ \mathbf{m_v} \end{bmatrix}, \ \mathbf{C_{ZZ}} = \begin{bmatrix} \mathbf{Q} & \mathbf{N} \\ \mathbf{N}^{\mathsf{T}} & \mathbf{R} \end{bmatrix}.$$

VII. Example

This section applies the various algorithms presented in this paper into simulating a noisy electric circuit with noisy measurements. The electric circuit considered, adapted from [12], is depicted in Fig. 1. The thermal noise in the resistors R_1, R_2, and R_3 will be modeled according to the the Johnson-Nyquist noise model, which models a noisy resistor as a noise-free resistor in series with a voltage source w_i, whose power spectral density is white and is given by $v_{w_i}^2 = 4kTR\Delta f$, where k is Boltzmann's constant (1.38×10^{-23} JK^{-1}), T is the resistor's absolute temperature in Kelvins, R is the nominal resistance in Ohms, and Δf is the bandwidth of operation in Hertz. Hence, the process noise vector will be defined as $\mathbf{w} \triangleq \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^{\mathsf{T}}$. The state variables will be chosen to be the current through the inductor L and the voltage across the capacitor C, i.e. $x_1 \triangleq i_L$ and $x_2 \triangleq v_c$. The circuit has two inputs, the voltage source v_s and the current source i_s. The measurements made on the circuit are the voltages across R_2 and R_3. The measurements are corrupted by an additive Gaussian noise having a constant DC bias b and a spectral density s^2. Hence, the process and measurement noise statistics are characterized by $\mathbf{m_w} = \mathbf{0}$, $\mathbf{m_v} = \begin{bmatrix} b & b \end{bmatrix}^{\mathsf{T}}$, $\mathbf{Q} = \mathrm{diag}\left[v_{w_1}^2, v_{w_2}^2, v_{w_3}^2\right]$, $\mathbf{R} = \mathrm{diag}\left[s^2, s^2\right]$, and $\mathbf{N} = \mathbf{0}$.

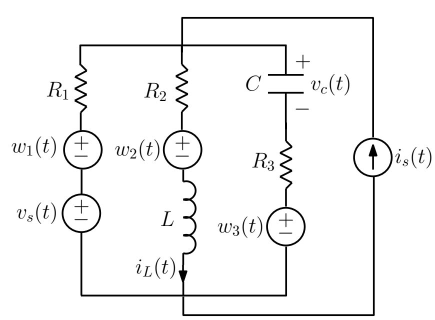

Fig. 1. Noisy electric circuit with noisy measurements

Using kirchhoff's voltage and current laws, the matrices governing the dynamics of the circuit can be expressed as

$$\mathbf{A}=\begin{bmatrix}-\frac{R_1R_2+R_1R_3+R_2R_3}{L(R_1+R_3)} & \frac{R_1}{L(R_1+R_3)}\\ \frac{-R_1}{C(R_1+R_3)} & \frac{-1}{C(R_1+R_3)}\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}\frac{R_1R_3}{R_1+R_3} & \frac{-R_1}{R_1+R_3}\\ R_2 & 0\end{bmatrix}$$

$$\mathbf{B}=\begin{bmatrix}\frac{R_3}{L(R_1+R_3)} & \frac{R_1R_3}{L(R_1+R_3)}\\ \frac{1}{C(R_1+R_3)} & \frac{-R_1}{C(R_1+R_3)}\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}\frac{R_1}{R_1+R_3} & \frac{R_1R_3}{R_1+R_3}\\ 0 & 0\end{bmatrix}$$

$$\mathbf{G}=\begin{bmatrix}\frac{R_3}{L(R_1+R_3)} & \frac{-1}{L} & \frac{R_1}{L(R_1+R_3)}\\ \frac{1}{C(R_1+R_3)} & 0 & \frac{-1}{C(R_1+R_3)}\end{bmatrix},\ \mathbf{H}=\begin{bmatrix}\frac{2R_1+R_3}{R_1+R_3} & 0 & \frac{-R_1}{R_1+R_3}\\ 0 & 1 & 0\end{bmatrix}.$$

The voltage and current sources were set to $v_s(t) = V_s\sin(2\pi f_1 t)$ and $i_s(t) = I_s\sin(2\pi f_2 t)$, respectively; where $V_s = 1\text{mV}$, $I_s = 0.1\text{mA}$, and $f_1 = f_2 = 0.1\text{MHz}$. The circuit components were set to $R_1 = R_2 = R_3 = 10\text{K}\Omega$, $C = 1\text{nF}$, and $L = 1\text{mH}$. The measurement noise DC bias and spectral density were set to $b = 1\mu\text{V}$ and $s = 1\text{pV}$, respectively.

The SDS was simulated via the EM scheme discussed in Section III, with $T = 100\text{ns}$. Then, the CT SDS was discretized at the same sampling period through the method presented in Section IV, with the DT matrices computed through the algorithms outlined in Section V. The process and measurement noise samples were generated according to the method discussed in Section VI. Fig. 2-3 and Fig. 4 illustrate a sample path of the CT and DT system states and measurements, respectively. Fig. 5 plots the mean-squared error (MSE) between the CT and DT system state x_1 for 100 Monte Carlo simulation runs. Similar results were noted with respect to the MSE between the CT and DT system state x_2 and measurements y_1 and y_2. The MSE for x_2, y_1, and y_2 were of the order of 4×10^{-7}, 1×10^{-3}, and 3×10^{-3}, respectively. It's noteworthy that the DT equivalent system offers a very close approximation to its CT counterpart.

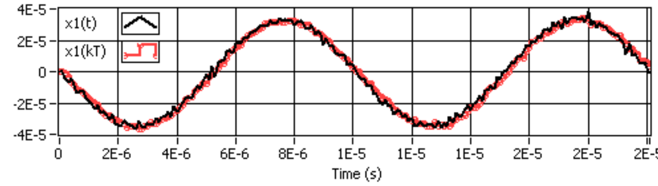

Fig. 2. $x_1(t)$ and $x_1(kT)$

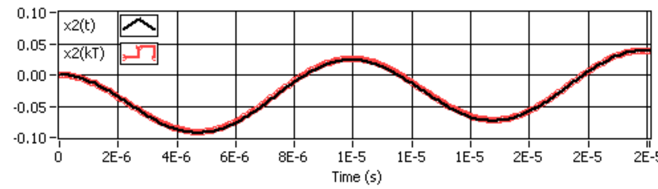

Fig. 3. $x_2(t)$ and $x_2(kT)$

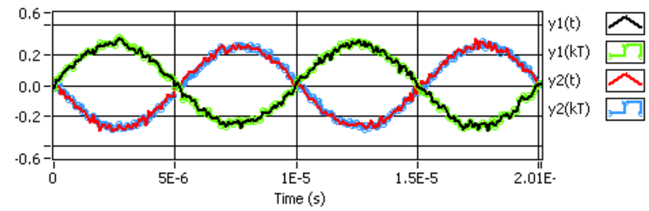

Fig. 4. $\mathbf{y}(t)$ and $\mathbf{y}(kT)$

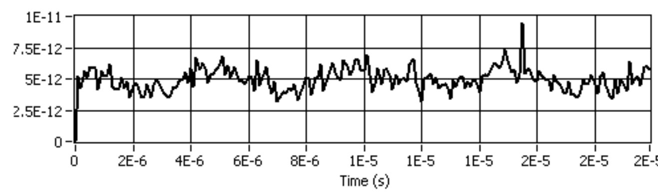

Fig. 5. MSE in x_1

VIII. Conclusion

This paper presented a self-contained treatment of the problem of numerical simulation of CT SDSs and their DT equivalents. In this respect, a general model for the SDS was adopted. The EM scheme for simulating SDEs was derived for the linear state-space representation of the SDS considered. It turns out that the derived EM scheme for the SDS model considered coincides with the more accurate Milstein scheme. The connection between the CT SDS and its discretization was established, and numerical algorithms were derived for computing the various system matrices and noise statistics. Moreover, a technique for generating non-zero mean, correlated process and measurement noise vectors was presented. Finally, the various algorithms discussed in the paper were applied through an example, which simulates the CT dynamics governing a noisy circuit with noisy measurements and the DT equivalent model.

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