

Discretization of MIMO Systems with Nonuniform Input and Output Fractional Time Delays

Zaher M. Kassas and Ricardo Dunia

Abstract—Input and output time delays in continuous-time state-space systems are treated separately as their effects are encountered before and after the state dynamics. Existing discretization techniques for such systems usually consider the delays to be integer multiples of the sampling time. This work develops a discretization procedure for multi-variable state-space systems with nonuniform input and/or output delays, in which such delays are non-integer multiples of the sampling time. A heat exchanger process example is used to illustrate the advantage of using this technique compared to the discretization approach of rounding the delays to the closest integer multiple of the sampling time. A significant improvement towards the continuous-time response was noted when considering the fractional delays into the discretized model. This discretization improvement is expected to benefit the design and performance of model-based controllers.

I. INTRODUCTION

Process measurements at the current sample time tend to reflect process values a few samples in the past. Time delays commonly affect measured variables in manufacturing facilities. Process instrumentation like thermocouples, pressure gauges, flow compensators, convertors, and transducers do not respond instantaneously to signal changes. Manipulated variables, known as process inputs, are also delayed from the moment the signal is computed or generated, to the time they impact the process in study. Such delays play a detrimental role in process control.

The control of systems with time delays is generally difficult in both theory and practice [1]. Time delays often put severe restrictions on achievable feedback performance [2]. Continuous-time linear systems with time delays are infinite-dimensional systems. The discrete-equivalent system is, however, finite dimensional as only the states at the sampling instants are of interest [3], [4]. The order reduction from infinite to finite dimensional makes the discretization of delayed systems not only desirable from a digital implementation perspective, but also from a simplified analysis and synthesis perspective as well. However, such a desirable dimensional reduction may increase the model mismatch between the continuous-time delayed system and its discrete-equivalent if the time delays happen to be non-integer multiples of the sampling time.

Discretization techniques for delayed continuous-time systems can be found in the literature [5], [6], [7]. However, these techniques often assume the time delay to be an integer multiple of the sampling time. Some approximations have

been proposed for delayed systems with time delays that are non-integer multiples of the sampling time, i.e. systems with fractional delays. Among them are the use of Taylor-series expansions, and to round the time delay to the closest integer multiple of the sampling time [8]. Nevertheless, these approximations incorporate model mismatches, which prevent the discrete-time model response from reaching the actual continuous-time response at the sampling instants.

Transfer functions of systems with fractional time delays have been thoroughly analyzed through the modified z -transform [9], [10], [11]. However, transfer functions characterize the input-output behavior of a system, while the state information is suppressed. Single-input single-output (SISO) continuous-time transfer functions with time delay(s) have the structure $G(s) = e^{-\lambda s}H(s)$, where $H(s)$ is the delay-free transfer function, and λ is the time delay. This structure shows that the discrete-equivalent transfer function, $G(z)$, will be the same regardless if such a delay corresponds to the input signal or the output signal [5]. This is not the case, however, for delayed state-space systems, in which there is a remarkable difference between input and output delays. State-space systems with fractional time delays have not been analyzed to the same extent as transfer functions. State variables often have physical interpretations, and it is desirable to obtain a discrete-equivalent state-space representation of the continuous-time system. Discretization of SISO linear time-invariant (LTI) systems with a fractional input time delay has been considered in [5], [6], [12]. Additionally, discretization of systems with “inner” time delays has been considered in [3], [4], [13]. However, the general problem of discretizing multiple-input multiple-output (MIMO) systems with nonuniform input and output fractional time delays has not been addressed in the literature. In this respect, it is meant by nonuniform delays that the different input and output signals are delayed by nonidentical values. This work derives an algorithm for obtaining the discrete-equivalent of a MIMO LTI continuous-time state-space system with nonuniform input and/or output fractional time delays.

This paper is organized in the following manner: Section II derives the algorithm for discretizing continuous-time systems with fractional delays by considering the cases of nonuniform input, output, and simultaneous input and output delays. These three cases account for all possible scenarios in practice. This discretization technique is applied to a heat exchanger process simulation example in Section III. Finally, Section IV presents conclusions and future work.

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Zaher M. Kassas and Ricardo Dunia are with National Instruments, Control Design & Simulation R&D, 11500 N. Mopac Expwy., Austin, TX 78759, USA. zkassas@ieee.org, rick.dunia@ni.com

II. SYSTEMS WITH NONUNIFORM INPUT AND OUTPUT FRACTIONAL TIME DELAYS

Discretization techniques for systems with nonuniform input and/or output time delays that are non-integer multiples of the sampling time are derived in this section. All the discretizations will assume a zero-order hold (ZOH), which implies that the input signal will take piecewise constant values within the sampling time T , i.e.

$$\underline{u}(t) = \underline{u}(kT), \quad kT \leq t < (k+1)T.$$

The input signals, $\underline{u}(t) \in \mathbb{R}^r$, are assumed to be delayed by the vector $\underline{\theta} \in \mathbb{R}^r$, where we write $\underline{u}(t - \underline{\theta})$, which will be interpreted as

$$\underline{u}(t - \underline{\theta}) \triangleq [u_1(t - \theta_1), u_2(t - \theta_2), \dots, u_r(t - \theta_r)]^T,$$

whereas the output signals, $\underline{y}(t) \in \mathbb{R}^m$, are delayed by $\underline{\phi} \in \mathbb{R}^m$, where we write $\underline{y}(t - \underline{\phi})$, which will be interpreted as

$$\underline{y}(t - \underline{\phi}) \triangleq [y_1(t - \phi_1), y_2(t - \phi_2), \dots, y_m(t - \phi_m)]^T.$$

Additionally, $\underline{\theta}$ and $\underline{\phi}$ are assumed to have values that are non-integer multiples of the sampling time.

The next two subsections will describe the discretization of continuous-time systems, where fractional time delays could be solely in the inputs or in the outputs. The general case where input and output fractional time delays coexist in the same system is derived in the third subsection.

A. Systems with Input Delays

An LTI system with nonuniform input delays can be expressed as

$$\dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t - \underline{\theta}) \quad (1)$$

$$\underline{y}(t) = \mathbf{C}\underline{x}(t) + \mathbf{D}\underline{u}(t - \underline{\theta}), \quad (2)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times r}$, $\mathbf{C} \in \mathbb{R}^{m \times n}$, $\mathbf{D} \in \mathbb{R}^{m \times r}$, and $\underline{\theta} \in \mathbb{R}^r$. The system matrices \mathbf{B} , \mathbf{C} , and \mathbf{D} will be interpreted as

$$\mathbf{B} = [\underline{b}_1 \mid \underline{b}_2 \mid \dots \mid \underline{b}_r], \quad \mathbf{C} = \begin{bmatrix} \underline{c}_1^T \\ \underline{c}_2^T \\ \vdots \\ \underline{c}_m^T \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \underline{d}_1^T \\ \underline{d}_2^T \\ \vdots \\ \underline{d}_m^T \end{bmatrix}.$$

The vector $\underline{\theta}$ will be decomposed as

$$\underline{\theta} \triangleq (\underline{l} - \underline{p})T, \quad (3)$$

where $l_j \in \mathbb{N}$, $p_j \in [0, 1)$, $j = 1, 2, \dots, r$, and T is the sampling time. Without loss of generality, we will only consider the case where $l_j \equiv 1$. If the j^{th} input signal happens to have a delay such that $l_j > 1$, then the residual $(l_j - 1)$ delay can be handled after the discrete-equivalent model is derived. This is done by introducing $(l_j - 1)$ state variables to account for these delays, which are essentially integer multiples of the sampling time [5].

The solution for the state dynamics in (1) is given by

$$\underline{x}(t) = e^{\mathbf{A}(t-t_0)}\underline{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\underline{u}(\tau - \underline{\theta})d\tau. \quad (4)$$

Letting $t_0 = kT$ and $t = (k+1)T$ in (4) and using the change of variables $\eta = (k+1)T - \tau$ along with (3) yields

$$\underline{x}[(k+1)T] = e^{\mathbf{A}T}\underline{x}(kT) + \underline{\alpha}, \quad (5)$$

where

$$\underline{\alpha} = \underline{\alpha}_1 + \underline{\alpha}_2 + \dots + \underline{\alpha}_r, \\ \underline{\alpha}_j = \int_0^T e^{\mathbf{A}\eta}\underline{b}_j u_j(kT + p_jT - \eta)d\eta, \quad j = 1, 2, \dots, r. \quad (6)$$

Fig. 1 illustrates that in the interval $\eta \in [0, p_jT)$, the input signal takes a piecewise constant value of $u(kT)$, whereas in the interval $\eta \in [p_jT, T)$ it takes a piecewise constant value of $u(kT - T)$. Therefore, (6) can be rewritten as

$$\underline{\alpha}_j = \underline{\psi}_j(p_jT) u_j(kT) + \underline{\Phi}(p_jT)\underline{\psi}_j(T - p_jT) u_j(kT - T),$$

where the following notation has been introduced

$$\underline{\Phi}(t) \triangleq e^{\mathbf{A}t} \\ \underline{\psi}_j(t) \triangleq \int_0^t e^{\mathbf{A}\tau}d\tau \underline{b}_j.$$

Consequently, substituting for $\underline{\alpha}$ in (5) yields

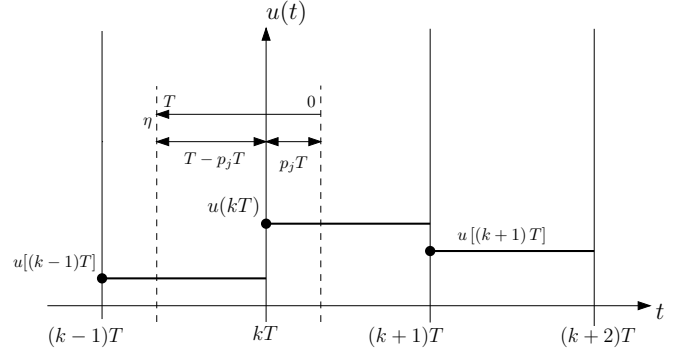


Fig. 1. Effect of Input Fractional Time Delay

$$\underline{x}[(k+1)T] = \underline{\Phi}(T)\underline{x}(kT) + \underline{\Omega}\underline{u}(kT) + \underline{\Sigma}\underline{u}(kT - T),$$

where

$$\underline{\Omega} = [\underline{\psi}_1(p_1T) \mid \dots \mid \underline{\psi}_r(p_rT)]$$

$$\underline{\Sigma} = [\underline{\Phi}(p_1T)\underline{\psi}_1(T - p_1T) \mid \dots \mid \underline{\Phi}(p_rT)\underline{\psi}_r(T - p_rT)].$$

Finally, defining the new state $\underline{\xi}(kT) \triangleq \underline{u}(kT - T)$, we get the discrete-equivalent for the system in (1) and (2) in the following compact form

$$\begin{bmatrix} \underline{x}[(k+1)T] \\ \underline{\xi}[(k+1)T] \end{bmatrix} = \begin{bmatrix} \underline{\Phi} & \underline{\Sigma} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x}(kT) \\ \underline{\xi}(kT) \end{bmatrix} + \begin{bmatrix} \underline{\Omega} \\ \mathbf{I} \end{bmatrix} \underline{u}(kT)$$

$$\underline{y}(kT) = [\mathbf{C} \quad \mathbf{D}] \begin{bmatrix} \underline{x}(kT) \\ \underline{\xi}(kT) \end{bmatrix} + [\mathbf{0}] \underline{u}(kT),$$

where \mathbf{I} is the identity matrix, and $\mathbf{0}$ is a matrix of zeros. The dependency of $\underline{\Phi}(T)$ on T has been dropped in the above equation for simplicity of notation.

B. Systems with Output Delays

An LTI system with nonuniform output delays can be expressed as

$$\dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t) \quad (7)$$

$$\underline{y}(t + \underline{\phi}) = \mathbf{C}\underline{x}(t) + \mathbf{D}\underline{u}(t), \quad (8)$$

where $\underline{\phi} \in \mathbb{R}^m$. The state equation in (7) can be readily discretized to get

$$\underline{x}[(k+1)T] = \Phi(T)\underline{x}(kT) + \Psi(T)\underline{u}(kT), \quad (9)$$

where

$$\Psi(t) \triangleq [\psi_1(t) \mid \psi_2(t) \mid \dots \mid \psi_r(t)].$$

Decomposing the delay vector $\underline{\phi}$ as

$$\underline{\phi} \triangleq (\underline{h} - \underline{q})T, \quad (10)$$

where $h_i \in \mathbb{N}$, $q_i \in [0, 1)$, and $i = 1, 2, \dots, m$, we may rewrite the output equation in (8) as

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix} = \begin{bmatrix} \underline{c}_1^T \underline{x}(t - \phi_1) \\ \underline{c}_2^T \underline{x}(t - \phi_2) \\ \vdots \\ \underline{c}_m^T \underline{x}(t - \phi_m) \end{bmatrix} + \begin{bmatrix} \underline{d}_1^T \underline{u}(t - \phi_1) \\ \underline{d}_2^T \underline{u}(t - \phi_2) \\ \vdots \\ \underline{d}_m^T \underline{u}(t - \phi_m) \end{bmatrix}. \quad (11)$$

As in the case of the nonuniform input delays, only the case where $h_i \equiv 1$, will be considered. If the i^{th} output signal happens to have a delay such that $h_i > 1$, then the residual $(h_i - 1)$ delay can be handled after the discrete-equivalent model is derived. Upon discretizing the output equation in (11) and using (10) we get

$$y_i(kT) = \underline{c}_i^T \underline{x}[kT - (1 - q_i)T] + \underline{d}_i^T \underline{u}[kT - (1 - q_i)T], \quad i = 1, 2, \dots, m. \quad (12)$$

The solution for the state dynamics in (7) is given by

$$\underline{x}(t) = e^{\mathbf{A}(t-t_0)}\underline{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\underline{u}(\tau) d\tau. \quad (13)$$

Letting $t_0 = (k-1)T$ and $t = kT - (1 - q_i)T$ in (13) we get

$$\begin{aligned} \underline{x}[(k-1+q_i)T] &= e^{\mathbf{A}q_i T} \underline{x}[(k-1)T] \\ &+ \int_{(k-1)T}^{(k-1+q_i)T} e^{\mathbf{A}[(k-1+q_i)T-\tau]} \mathbf{B}\underline{u}(\tau) d\tau. \end{aligned} \quad (14)$$

Noting that $\underline{u}(t) = \underline{u}[(k-1)T]$ for $(k-1)T \leq t < (k-1+q_i)T < kT$ for any $q_i \in [0, 1)$, and using the change of variable $\eta = (k-1+q_i)T - \tau$ in the integral in (14), we arrive at

$$\begin{aligned} \underline{x}[(k-1+q_i)T] &= \Phi(q_i T) \underline{x}[(k-1)T] \\ &+ \Psi(q_i T) \underline{u}[(k-1)T]. \end{aligned} \quad (15)$$

Next, we note that under the ZOH assumption, $\underline{u}[(k-1+q_i)T] = \underline{u}[(k-1)T]$ for any $q_i \in [0, 1)$. Utilizing this fact and (15), the output equation in (12) can be expressed as

$$\underline{y}(kT) = \mathbf{\Lambda} \underline{x}[(k-1)T] + \mathbf{\Upsilon} \underline{u}[(k-1)T],$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \underline{c}_1^T \Phi(q_1 T) \\ \underline{c}_2^T \Phi(q_2 T) \\ \vdots \\ \underline{c}_m^T \Phi(q_m T) \end{bmatrix}, \quad \mathbf{\Upsilon} = \begin{bmatrix} \underline{c}_1^T \Psi(q_1 T) + \underline{d}_1^T \\ \underline{c}_2^T \Psi(q_2 T) + \underline{d}_2^T \\ \vdots \\ \underline{c}_m^T \Psi(q_m T) + \underline{d}_m^T \end{bmatrix}.$$

Finally, the discrete-equivalent of the system in (7) and (8) can be represented compactly as

$$\begin{aligned} \begin{bmatrix} \underline{x}[(k+1)T] \\ \underline{y}[(k+1)T] \end{bmatrix} &= \begin{bmatrix} \Phi & \mathbf{0} \\ \mathbf{\Lambda} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x}(kT) \\ \underline{y}(kT) \end{bmatrix} \\ &+ \begin{bmatrix} \Psi \\ \mathbf{\Upsilon} \end{bmatrix} \underline{u}(kT) \\ \underline{y}(kT) &= [\mathbf{0} \quad \mathbf{I}] \begin{bmatrix} \underline{x}(kT) \\ \underline{y}(kT) \end{bmatrix} + [\mathbf{0}] \underline{u}(kT), \end{aligned}$$

where the dependency of $\Phi(T)$ and $\Psi(T)$ on T has been dropped in the above equations for simplicity of notation.

C. Systems with Input and Output Delays

An LTI system with nonuniform input and output delays can be expressed as

$$\dot{\underline{x}}(t) = \mathbf{A}\underline{x}(t) + \mathbf{B}\underline{u}(t - \underline{\theta}) \quad (16)$$

$$\underline{y}(t + \underline{\phi}) = \mathbf{C}\underline{x}(t) + \mathbf{D}\underline{u}(t - \underline{\theta}). \quad (17)$$

The delay vectors $\underline{\theta}$ and $\underline{\phi}$ will be defined according to (3) and (10), respectively. Without loss of generality, we will only consider the case where $l_j \equiv 1$ and $h_i \equiv 1$, where $j = 1, 2, \dots, r$ and $i = 1, 2, \dots, m$. If it happens that the j^{th} input signal and/or the i^{th} output signal has a delay such that $l_j > 1$ and/or $h_i > 1$, then the residual $(l_j - 1)$ and/or $(h_i - 1)$ delay can be dealt with independently once the discrete-equivalent model is achieved.

Since the system states are not affected by the output delays as they are only affected by the input delays, then the state equation discrete-equivalent is derived exactly as it was shown in Subsection II-A.

As for the output equation in (17), it can be shown that upon discretization and using (3) and (10) we get

$$\begin{aligned} y_i(kT) &= \underline{c}_i^T \underline{x}[kT - (1 - q_i)T] \\ &+ \underline{d}_i^T \underline{u}[kT - (\underline{1} - \underline{p})T - (1 - q_i)T], \end{aligned} \quad (18)$$

where $\underline{1} = [1, 1, \dots, 1]^T$ and $i = 1, 2, \dots, m$. To calculate the discrete state $\underline{x}[kT - (1 - q_i)T]$, we first let $t_0 = (k-1)T$ and $t = kT - (1 - q_i)T$ in (4), and then use the change of variable $\eta = kT - T + q_i T - \tau$ to give

$$\underline{x}[(k-1+q_i)T] = e^{\mathbf{A}q_i T} \underline{x}[(k-1)T] + \underline{\beta}_i, \quad (19)$$

where

$$\underline{\beta}_i = \int_0^{q_i T} e^{\mathbf{A}\eta} \mathbf{B} \underline{u}[kT - T + q_i T - (\underline{1} - \underline{p})T - \eta] d\eta. \quad (20)$$

Expanding (20) we find that

$$\begin{aligned} \underline{\beta}_i &= \underline{\beta}_{i,1} + \underline{\beta}_{i,2} + \dots + \underline{\beta}_{i,r}, \\ \underline{\beta}_{i,j} &= \int_0^{q_i T} e^{\mathbf{A}\eta} \underline{b}_j u_j [(k-2+q_i+p_j)T - \eta] d\eta, \end{aligned}$$

where $j = 1, 2, \dots, r$ in the above equation. The nature of the above integral can be understood with the aid of Figures 2 and 3, where Fig. 2 depicts the case where $0 \leq p_j + q_i < 1$, whereas Fig. 3 depicts the case where $1 \leq p_j + q_i < 2$. For the former case, we can note that in the interval $\eta \in [0, q_i T)$, the input signal takes a piecewise constant value of $u[(k-2)T]$. However, in the latter case, the input signal takes a piecewise constant value of $u[(k-1)T]$ in the interval $\eta \in [0, (p_j + q_i - 1)T)$ and then switches to the piecewise constant value of $u[(k-2)T]$ in the interval $\eta \in [(p_j + q_i - 1)T, q_i T)$. Hence, $\underline{\beta}_{i,j}$ reduces to

$$\underline{\beta}_{i,j} = \begin{cases} \underline{\psi}_j(q_i T) u_j[(k-2)T], & 0 \leq q_i + p_j < 1; \\ \underline{\psi}_j[(q_i + p_j - 1)T] u_j[(k-1)T] + \mathbf{\Phi}[(q_i + p_j - 1)T] \underline{\psi}_j[(1 - p_j)T], & 1 \leq q_i + p_j < 2, \\ u_j[(k-2)T], & 1 \leq q_i + p_j < 2, \end{cases}$$

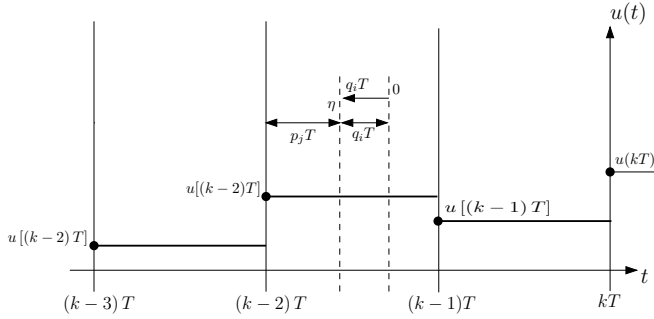


Fig. 2. Effect of Fractional Input and Output Time Delays with $0 \leq p_j + q_i < 1$

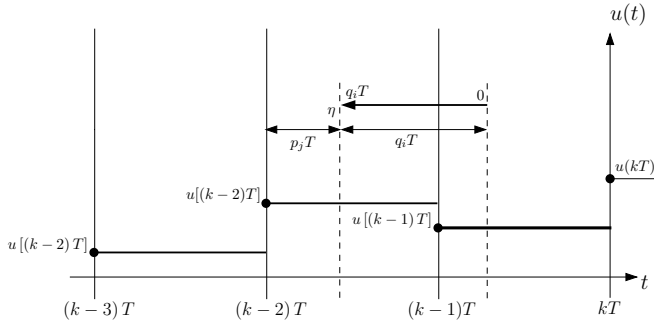


Fig. 3. Effect of Fractional Input and Output Time Delays with $1 \leq p_j + q_i < 2$

Therefore, (19) can be expressed compactly as

$$\underline{x}[(k-1+q_i)T] = \mathbf{\Phi}(q_i T) \underline{x}[(k-1)T] + \mathbf{\Delta}_i \underline{u}[(k-1)T] + \mathbf{\Gamma}_i \underline{u}[(k-2)T], \quad (21)$$

where

$$\mathbf{\Delta}_i = [\underline{\delta}_{i,1} \mid \underline{\delta}_{i,2} \mid \dots \mid \underline{\delta}_{i,r}]$$

$$\underline{\delta}_{i,j} = \begin{cases} 0 & 0 \leq q_i + p_j < 1; \\ \underline{\psi}_j[(q_i + p_j - 1)T], & 1 \leq q_i + p_j < 2, \end{cases}$$

$$\mathbf{\Gamma}_i = [\underline{\gamma}_{i,1} \mid \underline{\gamma}_{i,2} \mid \dots \mid \underline{\gamma}_{i,r}]$$

$$\underline{\gamma}_{i,j} = \begin{cases} \underline{\psi}_j(q_i T), & 0 \leq q_i + p_j < 1; \\ \mathbf{\Phi}[(q_i + p_j - 1)T], & 1 \leq q_i + p_j < 2, \\ \underline{\psi}_j[(1 - p_j)T], & 1 \leq q_i + p_j < 2, \end{cases}$$

where in the above equations $j = 1, 2, \dots, r$ and $i = 1, 2, \dots, m$.

To calculate $u_j[kT - (1 - p_j)T - (1 - q_i)T]$, we note that under the ZOH assumption we have

$$u_j[(k-2+p_j+q_i)T] = \begin{cases} u_j[(k-2)T], & 0 \leq q_i + p_j < 1; \\ u_j[(k-1)T], & 1 \leq q_i + p_j < 2. \end{cases}$$

Therefore, the input vector $\underline{u}[kT - (1 - p)T - (1 - q_i)T]$ can be represented compactly as

$$\underline{u}[kT - (1 - p)T - (1 - q_i)T] = \mathbf{V}_i \underline{u}[(k-1)T] + \mathbf{W}_i \underline{u}[(k-2)T], \quad (22)$$

where

$$\mathbf{V}_i = \text{diag}[v_{i,11}, v_{i,22}, \dots, v_{i,rr}]$$

$$v_{i,jj} = \begin{cases} 0, & 0 \leq q_i + p_j < 1; \\ 1, & 1 \leq q_i + p_j < 2. \end{cases}$$

and $\mathbf{W}_i = \mathbf{I} - \mathbf{V}_i$. Substituting the expressions derived in (21) and (22) in (18) we get

$$\underline{y}(kT) = \mathbf{\Lambda} \underline{x}[(k-1)T] + \mathbf{\Pi} \underline{u}[(k-1)T] + \mathbf{\Xi} \underline{u}[(k-2)T], \quad (23)$$

where

$$\mathbf{\Pi} = \begin{bmatrix} \underline{c}_1^T \mathbf{\Delta}_1 + \underline{d}_1^T \mathbf{V}_1 \\ \underline{c}_2^T \mathbf{\Delta}_2 + \underline{d}_2^T \mathbf{V}_2 \\ \vdots \\ \underline{c}_m^T \mathbf{\Delta}_m + \underline{d}_m^T \mathbf{V}_m \end{bmatrix}, \quad \mathbf{\Xi} = \begin{bmatrix} \underline{c}_1^T \mathbf{\Gamma}_1 + \underline{d}_1^T \mathbf{W}_1 \\ \underline{c}_2^T \mathbf{\Gamma}_2 + \underline{d}_2^T \mathbf{W}_2 \\ \vdots \\ \underline{c}_m^T \mathbf{\Gamma}_m + \underline{d}_m^T \mathbf{W}_m \end{bmatrix}.$$

Finally, the discrete-equivalent of the system in (16) and (17) can be represented compactly as

$$\begin{bmatrix} \underline{x}[(k+1)T] \\ \underline{\xi}[(k+1)T] \\ \underline{y}[(k+1)T] \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{\Lambda} & \mathbf{\Xi} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \underline{x}(kT) \\ \underline{\xi}(kT) \\ \underline{y}(kT) \end{bmatrix} + \begin{bmatrix} \mathbf{\Omega} \\ \mathbf{I} \\ \mathbf{\Pi} \end{bmatrix} \underline{u}(kT)$$

$$\underline{y}(kT) = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{I}] \begin{bmatrix} \underline{x}(kT) \\ \underline{\xi}(kT) \\ \underline{y}(kT) \end{bmatrix} + [\mathbf{0}] \underline{u}(kT),$$

where the dependency of $\mathbf{\Phi}(T)$ on T has been dropped in the above equation for simplicity of notation.

III. EXAMPLE

This section illustrates the application of the algorithm derived in Section II into discretizing a heat exchanger process with input and output fractional time delays. Two sets of single shell heat exchangers filled with water are placed in parallel and cooled by a liquid saturated refrigerant flowing through a coil system, as it is illustrated in Fig. 4. The saturated vapor generated in the coil system is separated from the liquid phase in the stages $S1$ and $S2$, both of neglected volumes. This vapor, withdrawn in $S1$ and $S2$, reduces the refrigerant mass flow rate along the cooling system, and only the saturated liquid portion is used for cooling purposes. Table I provides the fluid properties and equipment dimensions.

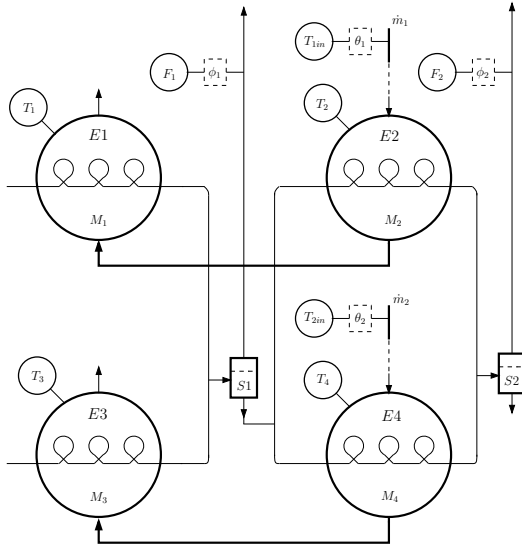


Fig. 4. Heat exchanger process with nonuniform input and output fractional time delays. The delayed inlet temperatures $T_{1in}(t - \theta_1)$ and $T_{2in}(t - \theta_2)$ represent the system inputs, while vapor flows $F_1(t + \phi_1)$ and $F_2(t + \phi_2)$ are the measured outputs.

TABLE I

FLUID PROPERTIES AND EQUIPMENT DIMENSIONS FOR THE HEAT EXCHANGER PROCESS

| | | |
|--------------|------------------------|-------------------------------------|
| C_p | 4.217KJ/KgK | Water specific heat |
| h_{lv} | 850KJ/Kg | Refrigerant heat of vaporization |
| $T_s(0)$ | 40°C | Initial temperature in E_s |
| $T_{jin}(0)$ | 40°C | Initial water inlet temperature j |
| T_c | 40°C | Refrigerant temperature |
| \dot{m}_j | 1Kg/s | Water mass flow j |
| M_s | 50Kg | Mass of water in E_s |
| hA | 8Kj/Kg | Overall surface heat transfer |
| V_1 | $3e^{-3}m^3$ | Inlet water pipe volume 1 |
| V_2 | $5e^{-3}m^3$ | Inlet water pipe volume 2 |
| ρ | 1000 Kg/m ³ | Water density |

The temperature of the refrigerant remains constant at T_c as the liquid is saturated, and the energy exchanged with water is used to vaporize a small portion of the refrigerant fluid. The energy balance around each heat exchanger is

given by

$$\begin{aligned} M_1 C_p \frac{dT_1}{dt} &= \dot{m}_1 C_p [T_2(t) - T_1(t)] - Q_1(t) \\ M_2 C_p \frac{dT_2}{dt} &= \dot{m}_1 C_p [T_{1in}(t - \theta_1) - T_2(t)] - Q_2(t) \\ M_3 C_p \frac{dT_3}{dt} &= \dot{m}_2 C_p [T_4(t) - T_3(t)] - Q_3(t) \\ M_4 C_p \frac{dT_4}{dt} &= \dot{m}_2 C_p [T_{2in}(t - \theta_2) - T_4(t)] - Q_4(t), \end{aligned}$$

where Q_s represents the heat exchanged in E_s , which is estimated using the overall surface heat transfer coefficient hA and the refrigerant temperature T_c , such that

$$Q_s(t) = hA[T_s(t) - T_c(t)], \quad s = 1, \dots, 4.$$

The input time delays θ_1 and θ_2 in the inlet water temperatures are due to the transportation time, such that

$$\theta_j = \frac{V_j \rho}{\dot{m}_j}, \quad j = 1, 2.$$

The measured output flow F_i is expressed as

$$F_i(t + \phi_i) = [Q_i(t) + Q_{i+2}(t)]/h_{lv} \quad i = 1, 2,$$

where h_{lv} is the refrigerant heat of vaporization. These outputs are delayed by $\phi_1 = 2.2$ sec and $\phi_2 = 3.8$ sec respectively, due to fluid property compensators. The individual heat exchanger energy balances can be expressed in terms of deviation variables to define the following LTI system matrices defined in (16) and (17) as

$$\mathbf{A} = \begin{bmatrix} -\frac{(1+\kappa_1)}{\tau_1} & \frac{1}{\tau_1} & 0 & 0 \\ 0 & -\frac{(1+\kappa_1)}{\tau_2} & 0 & 0 \\ 0 & 0 & -\frac{(1+\kappa_2)}{\tau_3} & \frac{1}{\tau_3} \\ 0 & 0 & 0 & -\frac{(1+\kappa_2)}{\tau_4} \end{bmatrix}$$

$$\mathbf{B}^T = \begin{bmatrix} 0 & \frac{1}{\tau_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\tau_4} \end{bmatrix}, \quad \mathbf{C} = \mu \cdot \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = [\mathbf{0}],$$

where $\kappa_j \triangleq \frac{\nu_j}{1+\nu_j}$, $\nu_j \triangleq \frac{hA}{\dot{m}_j C_p}$, $\mu \triangleq \frac{hA}{h_{lv}}$, and $\tau_s \triangleq M_s/\dot{m}_j$.

The LabVIEW Control Design Toolkit and Simulation Module were used to define the system dynamics with the appropriate input and output delays. The continuous-time system response is compared against the response of two discrete-time systems with different ZOH discretized delay assumptions. The first discrete-time system is obtained by rounding the fractional time delays for inputs and outputs to the closest integer multiple of the sampling time. The second discrete-time system is obtained by applying the ZOH discretization method for systems with fractional time delays, which was derived in Section II.

Figures 5 and 6 illustrate the output responses to step inputs of 5°C and -5°C on T_{1in} and T_{2in} at 1 sec and 10 sec of simulation, respectively, with a sampling time $T = 0.5$ sec. Note the response mismatch of the discrete-time system that rounds the delay to the nearest integer multiple of the sampling time. None of its response points lie on the continuous-time system response line. However, the response provided by the discrete-time system with fractional

delays matches exactly the continuous-time system response, as its points always remain on the actual response line at the precise sample instants.

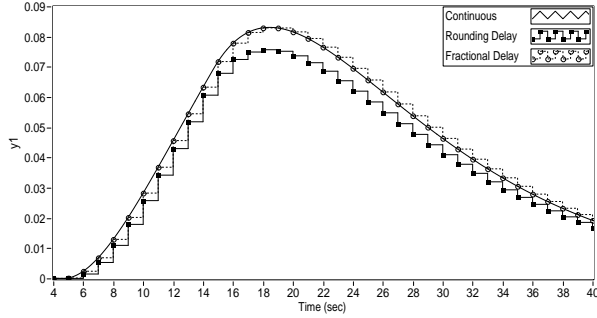


Fig. 5. Comparison between rounding and fractional delay ZOH discretization methods for y_1

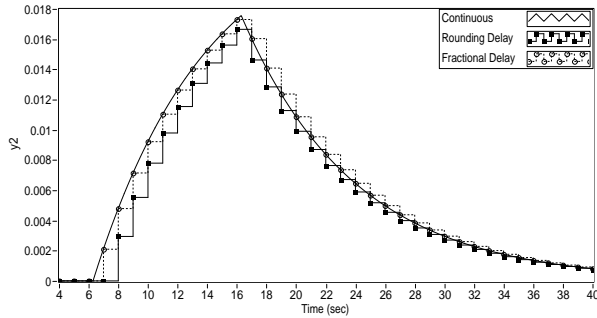


Fig. 6. Comparison between rounding and fractional delay ZOH discretization methods for y_2

Table II quantifies the detrimental effect of rounding input and output fractional time delays. The average relative percentage error along the simulation time, defined by

$$\varepsilon_i = \frac{1}{N} \sum_{k=1}^N \frac{|y_{ci}(kT) - y_{di}(kT)|}{y_{ci}(kT)} \times 100\%, \quad i = 1, 2, \quad N = 80,$$

determines the accuracy of the approximation relative to the magnitude of the actual response. Here, y_{ci} and y_{di} represent the continuous-time and discrete-time responses for output i , respectively. It can be noted from Table II that the average errors for the ZOH with rounding the delays are 13.7% and 12.2% for outputs y_1 and y_2 , respectively. No error is found for the fractional delay method, as this method accounts for the exact fractional delay into the discrete-time model.

TABLE II

RELATIVE PERCENTAGE ERRORS ALONG THE SIMULATION TIME FOR THE ROUNDING AND FRACTIONAL DISCRETIZATION METHODS

| ZOH Method | θ_1 | θ_2 | ϕ_1 | ϕ_2 | ε_1 | ε_2 |
|------------------|------------|------------|----------|----------|-----------------|-----------------|
| Fractional Delay | 1.5 | 2.5 | 2.2 | 3.8 | 0.0% | 0.0% |
| Rounding Delay | 1 | 2 | 2 | 4 | 13.7% | 12.2% |

IV. CONCLUSIONS AND FUTURE WORK

This work developed discretization techniques in which LTI MIMO state-space continuous-time systems with nonuniform input and/or output fractional time delays are discretized by using the ZOH method. Because ZOH is the only approximation made in this development, the response of the discrete-time system matches the continuous-time response at the times of sampling. Other discretization techniques for delayed systems tend to increase the continuous to discrete model mismatch in the presence of nonuniform fractional delay.

A heat exchanger process example is used to compare the approach of discretization by rounding to the closest integer multiple of the sampling time against the discretization method developed in this paper. An improvement of 13.7% and 12.2% for each output response towards the continuous-time response was found when considering fractional delays into the discrete model response.

The use of models that incorporate fractional time delays into model-based control applications will be investigated in future work. The extension of this technique to first-order hold (FOH) discretization is also contemplated for the near future.

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REFERENCES

- [1] C. Banyasz and L. Keviczky, "Robust stability limit of time-delay systems," in *Proc. IEEE American Control Conference'04*, Boston, MA, Jun. 2004, pp. 5428 – 5432.
- [2] J. Normey-Rico and E. Camacho, "Robust tuning of dead-time compensators for processes with an integrator and long dead-time," *IEEE Transactions on Automatic Control*, vol. 44, pp. 1597–1603, 1999.
- [3] B. Wittenmark, "Sampling of a system with a time delay," *IEEE Transactions on Automatic Control*, vol. 30, no. 5, pp. 507–510, May 1985.
- [4] —, "Sampling of a system with a time delay," in *Proc. IEEE 23rd Conference on Decision and Control'84*, Las Vegas, NV, Dec. 1984, pp. 1715 – 1720.
- [5] G. F. Franklin, J. D. Powell, and M. Workman, *Digital Control of Dynamic Systems Systems*, 3rd ed. Menlo Park, CA: Addison Wesley, 1997.
- [6] K. Astrom and B. Wittenmark, *Computer-Controlled Systems*, 3rd ed. Upper Saddle River, NJ: Prentice Hall, 1997.
- [7] T. Chen and B. Francis, *Optimal Sampled-Data Control Systems*. London, Great Britain: Springer-Verlag, 1995.
- [8] L. Hammarstrom and K. Gros, "Adaptation of optimal control theory to systems with time delays," *International Journal of Control*, vol. 32, no. 2, pp. 329–357.
- [9] E. I. Jury, *Theory and Application of the z-Transform Method*. New York: Wiley, 1964.
- [10] —, *Sampled Data Control Systems*. New York: Krieger, 1977.
- [11] K. Ogata, *Discrete-Time Control Systems*. Englewood Cliffs, NJ: Prentice Hall, 1987.
- [12] A. G. Kuznetsov and D. W. Clarke, "Simple numerical algorithms for continuous-to-discrete and discrete-to-continuous conversion of the systems with time delays," in *10th IFAC Symposium on System Identification Sysid'94*, vol. 3, Copenhagen, Denmark, July 1994, pp. 633–638.
- [13] B. Bernhardsson, "Sampling of state space systems with several time delays," in *Preprints IFAC 12th World Congress'93*, Sydney, Australia, 1993, pp. 361–364.